Proposed problems for the Magazine Contest

5^{th} Grade

CG:145 Prove that for any $n \in \mathbb{N}$ the natural number $a = 2^n + 3^n + 7^n$ cannot be a perfect square.

Liviu Smarandache, teacher, Craiova

CG:146 A natural number n, divided by the numbers 3,5,7,8 gives the remainders 1,3,5,6, respectively. Prove that n is even.

Gheorghe Tutulan, teacher, Galaţi

CG:147 Prove that the fraction $\frac{1\cdot 2\cdot 3\cdot ...\cdot 2008+1}{1\cdot 2\cdot 3\cdot ...\cdot 2009+1}$ is irreducible.

Dorina Andrei-Nicoară, teacher, Galaţi

CG:148 There are three pirate ships in Neverland. If a traveler who gets to a ship has an even number of gold coins with him, the captain gives him 3 more coins, but if he has an odd number of coins, the captain takes one, as well as half of the remaining coins. After Peter Pan comes aboard each of the three ships, he is left with 502 coins. At the beginning Peter Pan had an even number *n* of coins. Determine the digit sum of *n*.

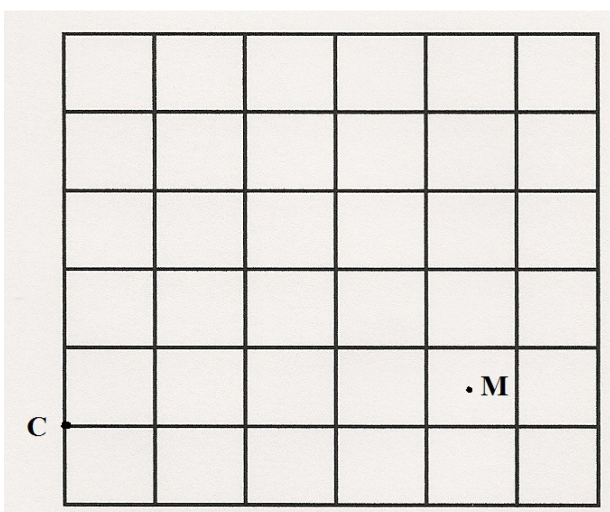
Romeo Zamfir, teacher, Galaţi

6^{th} Grade

CG:149 Prove that the equation $x^2 + 8 = y^4$ has no solution in $\mathbb{Z}\times\mathbb{Z}$.

Mihail Bencze, teacher, Braşov

CG:150 In a grid (see the figure) consider the point C and M the centre of a square. Draw the



line perpendicular to CM through M, by using only a straightedge.

Nicolae Ivăşchescu, teacher, Craiova

CG:151 In the quadrilateral $ABCD$ the angles A and C are right and $CB = CD$. Let us consider the point E such that $DE = AB$ and $D \in (AE)$. Prove that the triangle ACE is right.

Babis Stergiou, teacher, Greece

CG:152 Find the remainder of the division of the natural number $69^{2010} + 2010$ by 1587.

Mariana Coadă, teacher, Galaţi

7th Grade

CG:153 In the triangle ABC denote by O the intersection of the altitude $[AD]$ with the bisector $(BE$ of angle ABC, where $D \in (BC)$, $E \in (AC)$. Knowing that $[OA] = [OB]$, prove that the triangle ABE is isosceles if and only if $m(\sphericalangle C) = 45°$.

Cecilia Solomon, teacher, Galaţi

CG:154 Prove that $A = \dfrac{2^{2010} + 1}{2665}$ is a natural composite number.

Petre Bătrâneţu, teacher, Galaţi

CG:155 If $a, b, c \in \mathbb{R}$ and $a^2 + b^2 = c^2$, then write the expression $E = a^3 + b^3 + c^3$ as a product.

Babis Stergiou, teacher, Greece

CG:156 Find the first two digits of the number $\sqrt{1357911131517...20072009}$.

Mariana Coadă, teacher, Galaţi

8th Grade

CG:157 Consider $x, y, z \in \mathbb{R}_+^*$ so that $x \cdot y \cdot z = 1$. Prove that

$$\frac{1}{\sqrt{x} \cdot (y+z)} + \frac{1}{\sqrt{y} \cdot (z+x)} + \frac{1}{\sqrt{z} \cdot (x+y)} \le \frac{3}{2}.$$

Marius Cristian Manole, student, Galaţi

CG:158 Prove that there exists only one line segment with its endpoints on the graph of the function $f : \mathbb{R} \to \mathbb{R}$, $f(x) = x^2$, such that the coordinates of the endpoints are integers and the length of the line segment is expressed by a prime number.

Gheorghe Pădurariu, teacher, Galaţi

CG:159 Prove that all the quadratic equations with mutually distinct coefficients belonging to the set $\{2; 1 - 3\}$ have a common root.

Apostol Constantin, teacher, Rm. Sărat

CG:160 Determine the floor of the number $a = (2 - \sqrt{5})^{2009} + (3 - \sqrt{2})^{2009}$.

Romeo Zamfir, teacher, Galaţi

9^{th} Grade

CL:145 Consider $a, b, c > 0$ so that $a^2 + b^2 + c^2 = 1$. Prove that

$$a \cdot b \cdot c \cdot (a + b + c) + 2 \cdot a^2 \cdot b^2 + 2 \cdot b^2 \cdot c^2 + 2 \cdot c^2 \cdot a^2 \leq 1.$$

Cristinel Mortici

CL:146 Find all the real numbers x which can be uniquely written as $x = \frac{1}{a} + \frac{1}{b}$ with $0 < a \leq b$ and $a + b + 1 = 3 \cdot a \cdot b$.

Paul Cosma, student, Galaţi

CL:147 Consider the real numbers $a_1, a_2, ..., a_n > 1$ which satisfy the relation $\sum_{i=1}^{n} \frac{1}{a_i^2 - 1} = 1$. Prove that $\sum_{i=1}^{n} \frac{1}{a_i + 1} \leq \frac{1}{\sqrt{n+1} + 1}$.

Andrei Răzvan Băleanu, student, Motru

CL:148 Let $ABCD$ be a convex quadrilateral, $\{M\} = AC \cap BD$, $O_1, O_2, O_3, O_4, H_1, H_2, H_3, H_4$ being the centres of the excircles and the orthocentres of the triangles ABM, BCM, CDM, and DAM, respectively. Prove that $\overline{O_1O_4} = \overline{O_2O_3} \Leftrightarrow \overline{O_1H_2} + \overline{O_3H_4} = \overline{O_2H_1} + \overline{O_4H_3}$.

Sorin Pîrlea, teacher, Motătăi, Dolj

10^{th} Grade

CL:149 Let $n \geq 3$ be a natural number. Prove that, for all $a_1, a_2, a_3, ..., a_n \in \mathbb{R}_+^*$ the following inequality holds: $\frac{a_1 + a_2}{a_2 + a_3} + \sqrt{\frac{a_2 + a_3}{a_3 + a_4}} + \sqrt[3]{\frac{a_3 + a_4}{a_4 + a_5}} + ... + \sqrt[n]{\frac{a_n + a_1}{a_1 + a_2}} > \frac{n}{n-1}$. (For $n = 3$ we obtain the problem L:1041 from RMG 30, 2008, page 87.)

Andrei Crişan, student, and Sorin Ulmeanu

CL:150 Prove that in any triangle the following inequality holds: $(R + r)^3 \geq 3 \cdot \sqrt{3} \cdot p \cdot r^2$.

Nicuşor Minculete, teacher, Sfântu Gheorghe

CL:151 Solve in $\mathbb{R}^3$ the system: $\begin{cases} 5^x + 12^y = 13^x \\ 5^y + 12^z = 13^y \\ 5^z + 12^x = 13^z \end{cases}$.

Florin Antohe, teacher, Galaţi

CL:152 Find the real numbers $a_1, a_2, a_3, ..., a_n \in (1;+\infty)$ which satisfy the conditions $a_1 \cdot a_2 \cdot ... \cdot a_n = 2^n$ and $\log_{a_1}(a_1 - 1) + \log_{a_2}(a_2 - 1) + ... + \log_{a_n}(a_n - 1) = 0$.

Dan Seclăman and Lucian Ţuţescu, Craiova

11th Grade

CL:153 Let $OABC$ be a unit square. We construct the sequence $(A_n)_{n\geq 1}$ as following: A_1 is the midpoint of the line segment $[AB]$, A_2 is the midpoint of the line segment $[BC]$, A_3 is the midpoint of the line segment $[OC]$, A_4 is the midpoint of the line segment $[OA_1]$, A_5 is the midpoint of the line segment $[A_1A_2]$, A_6 is the midpoint of the line segment $[A_2A_3]$ and so on. If A_n has the coordinates (x_n, y_n) then check the convergence of the sequences $(x_n)_{n\geq 1}$ and $(y_n)_{n\geq 1}$.

Cerasela Momoiu, teacher, Galaţi

CL:154 Let $(x_n)_{n\geq 1}$ be a real number sequence and $\alpha \in \mathbb{R}^*$ with $x_1 \in \left(0;\frac{1}{\alpha}\right)$ and $x_{n+1} = x_n - \alpha \cdot x_n$ for every $n \in \mathbb{N}^*$. Calculate $l = \lim\limits_{n\to\infty}(n \cdot x_n)$ and $\lim\limits_{n\to\infty}\sqrt{n}\cdot(n \cdot x_n - l)$.

Mihai Dragoş Totolici, teacher, Galaţi

CL:155 Let $n \in \mathbb{N}^*$ and the invertible $n\times n$ complex number matrices A and B such that $B\cdot\left(I_n + A^{-1}\right) + I_n = A\cdot\left(I_n + B^{-1}\right)$. Prove that $\det(A - B) \neq 0$.

Ionuţ Ivănescu, teacher, Craiova

CL:156 Find the continuous functions $f:\mathbb{R}\to\mathbb{R}$ which satisfy the relation $f(x+y) - 3\cdot x^2\cdot y^2 = f(x) + f(y) + 2\cdot x\cdot y\cdot\left(x^2 + y^2\right)$, for all $x, y \in \mathbb{R}$.

Cristian Moanţă, teacher, Craiova

12th Grade

CL:157 Let $(G,\cdot)$ be a group and $a, b \in G$ so that $a^2 = e$ and $b = a^{-1}\cdot b^5\cdot a$. Prove that $b^{15} = a\cdot b^3\cdot a$.

Alfred Eckstein and Viorel Tudoran, Arad

CL:158 Find the continuous functions $f, g:\left(0;\frac{\pi}{2}\right)\to\mathbb{R}$ that satisfy the conditions $f(x) = -G(x)\cdot \mathrm{tg}^2 x$, $F(x) = g(x)\cdot \mathrm{tg}^2 x$ for all $x \in \left(0;\frac{\pi}{2}\right)$, where $F \in \int f(x)\,dx$ and $G \in \int g(x)\,dx$.

Florin Stănescu, teacher, Găeşti

CL:159 Let $(A,+,\cdot)$ be a commutative ring such that there exist the invertible elements $x, y \in A$ with $x + x^{-1} = 1 + y + y^{-1}$. Prove that $x - y$ is invertible in the ring A.

Laura Popescu, teacher, Craiova

CL:160 Let us consider the function $f:(0;+\infty)\to\mathbb{R}$, where $f(x)=\dfrac{x^2+x}{2\cdot x^3+3\cdot x^2+3\cdot x+1}$ for all $x\in(0;+\infty)$. Calculate $\max\int_{\frac{1}{2}}^{1} f(x)\,dx$.

Gheorghe Tutulan, teacher, Galaţi

Training Problems for the International Mathematics Olympiad

O:127 We say that a set has *dots* if it can be represented as a union of non-degenerate, disjoint circles.

a) If A and B are two distinct points on a sphere S, then prove that $S\setminus\{A;B\}$ has dots.

b) Let T be an open ball with centre O and radius $r>0$ (the set of points P with $OP<r$), and a point C so that $OC=r$. Prove that $T\cup\{C\}$ has dots.

c) Prove that the space has dots.

Paul Cosma, student, Galaţi

O:128 Consider a convex quadrilateral $ABCD$ with $AB=CB$ and $\sphericalangle ABC+2\cdot\sphericalangle CDA=\pi$, and let E be the midpoint of AC. Show that $\sphericalangle CDE\equiv\sphericalangle BDA$.

Paolo Leonetti

O:129 Let M be a set with exactly 2010 elements, which are consecutive integers. Prove that there do not exist two sets A and B with the properties $A\cup B=M$, $A\cap B=\varnothing$, and $\prod_{a\in A}a=\prod_{b\in B}b$.

Paul Cosma, student, Galaţi

O:130 Solve in $\mathbb{N}\times\mathbb{N}$ the equation $2+3^x=5^y$.

Vasile Popa, teacher, Galaţi